Marked and labelled Gushel-Mukai fourfolds

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Joint work with Emma Brakkee (arXiv:2002.04248)

Outline

1. Hodge theory for Gushel-Mukai fourfolds;
2. Associated K3 surface and main results;
3. Application to Fourier-Mukai partners.

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Gushel-Mukai fourfolds

We work over $\mathbb{C}$. Let V_5 be a 5-dimensional $\mathbb{C}$-vector space.

Consider:

- $\mathrm{Gr}(2, V_5) \subset \mathbb{P}(\bigwedge^2 V_5) \cong \mathbb{P}^9$ via Plücker embedding;
- the cone over the Grassmannian $\mathrm{Gr}(2, V_5)$ with vertex $\nu := \mathbb{P}(\mathbb{C})$
$$\mathrm{CGr}(2, V_5) \subset \mathbb{P}(\mathbb{C} \oplus \bigwedge^2 V_5) \cong \mathbb{P}^{10};$$
- $\mathbb{P}(W) \cong \mathbb{P}^8 \subset \mathbb{P}(\mathbb{C} \oplus \bigwedge^2 V_5)$;
- a quadric hypersurface $Q \subset \mathbb{P}(W)$.

Definition

A Gushel–Mukai (GM) fourfold is a smooth four-dimensional intersection
$$X = \mathrm{CGr}(2, V_5) \cap Q.$$

If $H \subset X$ is the hyperplane class, then $H^4 = 10$ and $K_X = -2H$. Thus X is a *Fano fourfold*.

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Motivations

GM fourfolds were firstly studied by:

- Debarre, Iliev, Manivel and Debarre, Kuznetsov from the Hodge-theoretical view point;
- Kuznetsov, Perry from the derived category view point.

Leitmotiv

Analogy with cubic fourfolds.

GM fourfolds are “related” to K3 surfaces and give rise to a rich hyperkähler geometry.

Our goal:

Explain better the connection with K3 surfaces on the level of period domains and moduli stacks/spaces $\rightsquigarrow$ arises a difference with cubic fourfolds.

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Interlude on lattices

Given a lattice L,

- $\operatorname{Disc} L := L^\vee / L$ is the discriminant group of L;
- the discriminant of L is the order of $\operatorname{Disc} L$;
- $\widetilde{O}(L) := \operatorname{Ker}(O(L) \to O(\operatorname{Disc} L))$ is the stable orthogonal group of L;
- given $m \neq 0 \in \mathbb{Z}$, $L(m)$ is the lattice L with the intersection form multiplied by m.

Examples

- $I_1 = (\mathbb{Z}, (1))$;
- $A_1 = I_1(2)$;
- $U = \left(\mathbb{Z}^{\oplus 2}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$ is the hyperbolic plane;
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Hodge diamond of X (Iliev, Manivel):

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Let $\gamma_X : X \to \mathrm{Gr}(2, V_5)$ be the linear projection from the vertex.
Remark: $\gamma_X^* : H^4(\mathrm{Gr}(2, V_5), \mathbb{Z}) = \langle \sigma_1^2, \sigma_{1,1} \rangle \hookrightarrow H^4(X, \mathbb{Z})$.

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The vanishing lattice of X is the sublattice
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Period domain and period map

The period domain of GM fourfolds is

$$\Omega(\Lambda_{00}) := \{w \in \mathbb{P}(\Lambda_{00} \otimes \mathbb{C}) : w \cdot w = 0, w \cdot \bar{w} < 0\}.$$

$\widetilde{\mathrm{O}}(\Lambda_{00})$ acts properly discontinuously on $\Omega(\Lambda_{00})$

$$\rightsquigarrow \qquad \mathcal{D} := \Omega(\Lambda_{00})/\widetilde{\mathrm{O}}(\Lambda_{00})$$

is an irreducible quasi-projective variety of dimension 20.

The period map is

$$p\colon \mathcal{M}_4 \to \mathcal{D}, \quad [X] \mapsto [\mathrm{H}^{3,1}(X)]$$

where $\mathcal{M}_4$ is the moduli stack of GM fourfolds.

Proposition (Debarre, Iliev, Manivel)

p is dominant as a map of stacks with smooth 4-dimensional fibers.

First difference with cubic fourfolds, whose period map is injective!
Torelli Theorem does not hold for GM fourfolds.

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$$\Omega(\Lambda_{00}) := \{w \in \mathbb{P}(\Lambda_{00} \otimes \mathbb{C}) : w \cdot w = 0, w \cdot \bar{w} < 0\}.$$

$\widetilde{\mathrm{O}}(\Lambda_{00})$ acts properly discontinuously on $\Omega(\Lambda_{00})$

$$\rightsquigarrow \qquad \mathcal{D} := \Omega(\Lambda_{00})/\widetilde{\mathrm{O}}(\Lambda_{00})$$

is an irreducible quasi-projective variety of dimension 20.

The period map is

$$p \colon \mathcal{M}_4 \to \mathcal{D}, \quad [X] \mapsto [\mathrm{H}^{3,1}(X)]$$

where $\mathcal{M}_4$ is the moduli stack of GM fourfolds.

Proposition (Debarre, Iliev, Manivel)

p is dominant as a map of stacks with smooth 4-dimensional fibers.

First difference with cubic fourfolds, whose period map is injective!
Torelli Theorem does not hold for GM fourfolds.

Special GM fourfolds

A very general GM fourfold X satisfies $\operatorname{rk} \mathrm{H}^{2,2}(X, \mathbb{Z}) = 2$.

Definition

A GM fourfold X is Hodge-special if $\operatorname{rk} H^{2,2}(X, \mathbb{Z}) \geq 3$.

Period points of Hodge-special GM fourfolds lie in codimension-1 Noether–Lefschetz loci in $\mathcal{D}$:

- For a primitive sublattice $L_d \subset \Lambda$ of rank 3 and discriminant d containing λ_1, λ_2, define

$$\Omega(L_d^{\perp}) := \mathbb{P}(L_d^{\perp} \otimes \mathbb{C}) \cap \Omega(\Lambda_{00})$$

where $L_d^{\perp}$ is the orthogonal complement of L_d in Λ.

- Let

$$\mathcal{D}_{L_d} \subset \mathcal{D}$$

be the image of $\Omega(L_d^{\perp})$ under the map $\Omega(\Lambda_{00}) \to \mathcal{D}$.

The period of any Hodge-special GM fourfold lies in $\mathcal{D}_{L_d}$ for some L_d.

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Properties (Debarre, Iliev, Manivel)

- $\mathcal{D}_{L_d} \neq \emptyset \Leftrightarrow d \equiv 0, 2, 4 \pmod 8$.
- Up to the action of $\widetilde{\mathrm{O}}(\Lambda_{00})$, $L_d \cong$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2k \end{pmatrix} \quad \text{if } d = 8k,$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2k \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2k \end{pmatrix} \quad \text{if } d = 2 + 8k,$$

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Stacks of Hodge-special GM fourfolds

Summary

Periods of Hodge-special GM fourfolds are contained in the union of

- *irreducible hypersurfaces* $\mathcal{D}_d \subset \mathcal{D}$ *for all* $d \equiv 0 \mod 4$;
- *the unions* $\mathcal{D}_d := \mathcal{D}'_d \cup \mathcal{D}''_d$ *of irreducible hypersurfaces for all* $d \equiv 2 \mod 8$ (*second difference with cubic fourfolds*).

We say X has discriminant d if $p(X) \in \mathcal{D}_d$.

Theorem (Debarre, Iliev, Manivel)

$\mathcal{D}_{L_d} \cap Im(p)$ *for* $d > 8$.

The moduli stack of GM fourfolds of discriminant d is

$$\mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}_d.$$

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Associated K3 surface

Definition (Debarre, Iliev, Manivel)

$X \in \mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}_d$ has a Hodge-associated degree-d polarized K3 surface (S, l) if there is a Hodge isometry $L_d^\perp \cong H^2(S, \mathbb{Z})_{\text{prim}}(-1)$.

$\Leftrightarrow$ $d \equiv 2, 4 \mod 8$ and $p \nmid d$ for every prime $p \equiv 3 \mod 4$. $(**)$

Motivation:

Conjecture (Players of GM fourfolds)

X is rational if and only if $X \in \mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}_d$ for d satisfying $(**)$.

Known examples of rational GM fourfolds are in

- $\mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}'_{10}$, $\mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}''_{10}$ (Debarre, Iliev, Manivel);
- $\mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}_{20}$ (Hoff, Staglianò in 2019);
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Relation on the stack level

If d satisfies $(**)$,

Question: what happens on the level of quotient domains and stacks?

- $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}} \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus I_1(-d) =: \Lambda_d$;
- $\Omega(\Lambda_d)/\widetilde{\mathrm{O}}(\Lambda_d)$ is the moduli space of quasi-polarized degree-d K3 surfaces.

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Marked and labelled GM fourfolds (à la Hassett)

Definition

A marked GM fourfold is a Hodge-special GM fourfold X together with a primitive embedding $\varphi\colon L_d \hookrightarrow \mathrm{H}^{2,2}(X,\mathbb{Z})$ preserving the classes λ_1 and λ_2. A labelled GM fourfold is a Hodge-special GM fourfold X together with a primitive sublattice $L_d \subset \mathrm{H}^{2,2}(X,\mathbb{Z})$.

Remark: Note that

$\{$ marked GM 4-folds $\}/_{\cong} \to \{$ labelled GM 4-folds $\}/_{\cong}$

$(X, \varphi\colon L_d \hookrightarrow \mathrm{H}^{2,2}(X,\mathbb{Z})) \mapsto (X, \varphi(L_d) \subset \mathrm{H}^{2,2}(X,\mathbb{Z}))$ is surjective, but need not to be injective.

Remember: $\widetilde{\mathrm{O}}(\Lambda_{00}) \cong \{g \in \mathrm{O}(\Lambda) : g|_{\langle\lambda_1,\lambda_2\rangle} = \mathrm{id}\} =: \Gamma$, so define

$$G(L_d) := \{g \in \Gamma : g(L_d) = L_d\},$$
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Marked and labelled GM fourfolds (à la Hassett)

Definition

A marked GM fourfold is a Hodge-special GM fourfold X together with a primitive embedding $\varphi\colon L_d \hookrightarrow \mathrm{H}^{2,2}(X,\mathbb{Z})$ preserving the classes λ_1 and λ_2. A labelled GM fourfold is a Hodge-special GM fourfold X together with a primitive sublattice $L_d \subset \mathrm{H}^{2,2}(X,\mathbb{Z})$.

Remark: Note that
$\{$ marked GM 4-folds $\}/_{\cong} \to \{$ labelled GM 4-folds $\}/_{\cong}$
$(X, \varphi\colon L_d \hookrightarrow \mathrm{H}^{2,2}(X,\mathbb{Z})) \mapsto (X, \varphi(L_d) \subset \mathrm{H}^{2,2}(X,\mathbb{Z}))$ is surjective, but need not to be injective.
Remember: $\widetilde{\mathrm{O}}(\Lambda_{00}) \cong \{g \in \mathrm{O}(\Lambda) : g|_{\langle\lambda_1,\lambda_2\rangle} = \mathrm{id}\} =: \Gamma$, so define

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Stacks of marked and labelled GM fourfolds

Remark $\Rightarrow$ $\mathcal{D}^{\mathrm{mar}}_{L_d} \twoheadrightarrow \mathcal{D}^{\mathrm{lab}}_{L_d}$

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$\mathcal{D}^{\bullet}_{d} := \mathcal{D}^{\bullet}_{L_d}$ if $d \equiv 0 \mod 4$.

When $d \equiv 2 \mod 8$, we have two embeddings $\mathcal{D}_{L_d} \xrightarrow{\cong} \mathcal{D}'_d \subset \mathcal{D}$ and $\mathcal{D}_{L_d} \xrightarrow{\cong} \mathcal{D}''_d \subset \mathcal{D}$; let $(\mathcal{D}'_d)^{\bullet}$ and $(\mathcal{D}''_d)^{\bullet}$ be the corresponding spaces $\mathcal{D}^{\bullet}_{L_d}$ over $\mathcal{D}'_d$ and $\mathcal{D}''_d$, respectively.

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Definition

The moduli stacks of marked and labelled GM fourfolds of discriminant d are $\mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}^{\mathrm{mar}}_d$ and $\mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}^{\mathrm{lab}}_d$.

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Main results

Theorem (Brakkee, P.)

The map $\mathcal{D}_{L_d}^{\text{mar}} \twoheadrightarrow \mathcal{D}_{L_d}^{\text{lab}}$ *is an isomorphism.*
As a consequence, $\mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}_d^{\text{mar}} \cong \mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}_d^{\text{lab}}$.

Corollary

For every d *satisfying* $(**)$, *there is a rational map*
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Indeed,

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\Omega(\Lambda_d(-1)) & \xrightarrow{\cong} & \Omega(L_d^{\perp}) & \hookrightarrow & \Omega(\Lambda_{00}) \\
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Comments

- $\mathcal{D}_d \dashrightarrow \Omega(\Lambda_d)/\widetilde{\mathrm{O}}(\Lambda_d)$ is birational for $d \equiv 0 \mod 4$, generically two-to-one if $d \equiv 2 \mod 8$.
- γ_d is not unique;
- Analogous statement for GM fourfolds having an associated twisted K3 surface:
  - X has associated twisted K3 surface $\Leftrightarrow d' = \prod_i p_i^{n_i}$ with $n_i \equiv 0 \mod 2$ for $p_i \equiv 3 \mod 4$ (P.);
  - use moduli spaces of polarized twisted K3 surfaces with fixed degree and order (Brakkee).
- Third difference with cubic fourfolds:
  - $\mathcal{D}^{\mathrm{mar}}_{L_d} \to \mathcal{D}^{\mathrm{lab}}_{L_d}$ is isomorphism or two-to-one cover (Hassett).
  - In the second case, there are "two" K3 surfaces, one is a moduli space of stable sheaves on the other (Brakkee).

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Derived category of GM fourfolds

Proposition (Kuznetsov, Perry)

$$\mathrm{D^b}(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{U}_X^*, \mathcal{O}_X(1), \mathcal{U}_X^*(1) \rangle$$

The Kuznetsov component of X is

$$\mathrm{Ku}(X) := \{E \in \mathrm{D}^b(X) : \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X(i), E) = 0, \\ \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{U}_X^*(i), E) = 0 \text{ for all } i = 0, 1\}.$$

Key property:
$\mathrm{Ku}(X)$ is a subcategory of K3 type: it has the same Serre functor and same Hochschild homology of $\mathrm{D}^b(S)$, where S is a K3 surface.

Definition

X has a homological associated K3 surface if $\mathrm{Ku}(X) \simeq \mathrm{D}^b(S)$ for a K3 surface S.

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Application to FM partners

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A GM fourfold X' is a Fourier–Mukai partner of X if there exists an exact equivalence $\mathrm{Ku}(X) \simeq \mathrm{Ku}(X')$ of Fourier–Mukai type.

Kuznetsov, Perry

Non-isomorphic GM fourfolds in the same fiber of the period map are FM partners. So Categorical Torelli Theorem does not hold for GM fourfolds.

Theorem (Brakkee, P.)

Let X be very general in $\mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}_d$ with d satisfying $(**)$. Let $\tau(d)$ be the number of distinct primes that divide $d/2$. Then when $d \equiv 4 \mod 8$ (resp. $d \equiv 2 \mod 8$), there are $2^{\tau(d)-1}$ (resp. $2^{\tau(d)}$) fibers of the period map such that, when non-empty, their elements are FM partners of X. Moreover, all FM partners of X are obtained in this way.

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Comments

- Analogous result for cubic fourfolds (P. and Fan, Lai 2020).
- We prove a similar statement in the case of associated twisted K3 surface.

For the proof we use:

- the rational map $\gamma_d : \mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}_d \dashrightarrow \Omega(\Lambda_d)/\widetilde{\mathrm{O}}(\Lambda_d)$;
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Fourth difference with cubic fourfolds, where these notions are equivalent (Addington, Thomas and Bayer, Lahoz, Macrì, Nuer, Perry, Stellari).

If X has a Hodge-associated K3 surfaces, then X has a homological associated K3 surface, but there are counterexamples to the inverse statement (P. + Perry, P., Zhao).

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Comments

- Analogous result for cubic fourfolds (P. and Fan, Lai 2020).
- We prove a similar statement in the case of associated twisted K3 surface.

For the proof we use:

- the rational map $\gamma_d : \mathcal{M}_4 \times_{\mathcal{D}} \mathcal{D}_d \dashrightarrow \Omega(\Lambda_d)/\widetilde{\mathrm{O}}(\Lambda_d)$;
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- Suggestion of Macrì: try to use stability conditions, recovering the GM fourfold inside a moduli space of Bridgeland stable objects in $\mathrm{Ku}(X)$.

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